REACHING GENERALIZED CRITICAL VALUES OF A POLYNOMIAL

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Abstract. Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We give an algorithm to compute the set of generalized critical values. The algorithm uses a finite dimensional space of rational arcs along which we can reach all generalized critical values of f.

1. Introduction.

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Over forty years ago R. Thom proved that f is a $C^\infty-$fibration outside a finite set, the smallest such a set is called *the bifurcation set of* f, we denote it by $B(f)$. In a natural way appears a fundamental question: how to determine the set $B(f)$.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$ - the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a large ball). To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values of* f

$$K_\infty(f) = \{y \in \mathbb{K} : \exists_{\mathbb{K}^n \ni x_l \to \infty}\ s.t.\ f(x_l) \to y\ and\ \|x_l\|\|df(x_l)\| \to 0\}.$$

If $c \notin K_\infty(f)$, then it is usual to say that f satisfies *Malgrange's condition* at c. It is proved ([16], [17]), that $B_\infty(f) \subset K_\infty(f)$. We call $K(f) = K_0(f) \cup K_\infty(f)$ *the set of generalized crtitical values of* f. Thus we have that in general $B(f) \subset K(f)$. In the case $\mathbb{K} = \mathbb{C}$ we gave in [9] an algorithm to compute the set $K(f)$.

In the real case, that is for a given real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ we can compute $K(f_{\mathbb{C}})$ the set of generalized critical values of $f_{\mathbb{C}}$ which stands for the complexification of f. However in general the set $K_\infty(f)$ of asymptotic critical values of f may be smaller than $\mathbb{R} \cap K_\infty(f_{\mathbb{C}})$. Precisely, it is possible (c.f. Example 4.1) that there exists a sequence $x_l \in \mathbb{C}^n \setminus \mathbb{R}^n$, $\|x_l\| \to \infty$ such that $f(x_l) \to y \in \mathbb{R}$ and $\|x_l\|\|df(x_l)\| \to 0$, but there is no sequence $x_l \in \mathbb{R}^n$ with this property.

To our best knowledge no method was known to detect algorithmically this situation.

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In the paper we propose another approach to the computation of generalized critical values which works both in the complex and in the real case. The main new idea is to use a finite dimensional space of rational arcs along which we can reach all asymptotic critical values.

Asymptotic and generalized critical values appear for instance in the problem of optimization of real polynomials, see e.g. papers of Hà and Pham [13], [14]. In fact, as they observed, if a polynomial f is bounded from below then $\inf f \in K(f)$. Numerical and complexity aspects of this approach to the optimization of polynomials were studied recently by M. Safey El Din, eg. [18], [19].

2. The complex case

We start with the following variant of the Puiseux Theorem:

Lemma 2.1. *Let $C \subset \mathbb{C}^n$ be a curve of degree d. Assume that $a \in (\mathbb{P}^n \setminus \mathbb{C}^n) \cap \overline{C}$ is a point at infinity of C. Let Γ be an irreducible component of the germ $\overline{C}_a$. Then there is an integer $s \le d$ and a real number $R > 0$, such that the Γ has a holomorphic parametrization of the type*

$$x = \sum_{-\infty \le i \le s} a_i t^i, \ |t| > R,$$

where $t \in \mathbb{C}$, $a_i \in \mathbb{C}^n$ and $\sum_{i>0} |a_i| > 0$.

Proof. Let $\overline{C} \setminus C = \{a, b_1, ..., b_r\}$. First choose the affine system of coordinates in $\mathbb{C}^n$ in a generic way. Let $a = (0 : a_1 : ... : a_n), b_j = (0 : b_{j1} : ...b_{jn}) \in \mathbb{P}^n$. Since our system of coordinates was generic we can assume that $a_i \neq 0$ for $i > 0$ and $b_{ji} \neq 0$ for $i, j > 0$. Choose a new projective system of coordinates, at which the new hyperplane at infinity is a hyperplane $H = \{x : x_1 = 0\}$. Take $y_1 = x_0/x_1$ and $y_i = x_i/x_0$ for $i = 2, ..., n$. Put $L = \{y \in \mathbb{C}^n : y_2 = y_3 = ... = y_n = 0\}$. By our construction we have $L \cap \overline{C} = \emptyset$. In particular the projection $\pi_L : \overline{C} \setminus H \to \mathbb{C}$ is finite. This means that there is a punctured disc $U = \{z \in \mathbb{C} : 0 < |z| < \delta\}$ such that the mapping

$$\rho : \Gamma \cap \pi_L^{-1}(U) \ni x \to \pi_L(x) \in U$$

is proper. We can also assume that the set $\Gamma' := \Gamma \cap \pi_L^{-1}(U)$ is smooth and ρ has no critical values on U. In particular ρ is a holomorphic covering of degree $s \le d$.

In particular the function $\rho^{-1} : U \ni z \to (z, h_2(z), ..., h_n(z)) \in \Gamma'$ is an $s-$valued holomorphic function. If we compose it with the mapping $z \to z^s$ we obtain a holomorphic function. Consequently the mapping $t \to (t^s, h_2(t^s), ..., h_n(t^s)) = (t^s, g_2(t), ..., g_n(t))$ is holomorphic. If we go back to the old coordinates we have the following parametrization of Γ :

$$t \to (1/t^s, g_2(t)/t^s, ..., g_n(t)/t^s),$$

where $0 < |t| < \delta$. Now exchange t by $1/t$ and put $R = 1/\delta$. □

Let $X \subset \mathbb{C}^m$ be a variety, recall that a mapping $F : X \to \mathbb{C}^m$ is *not proper* at a point $y \in \mathbb{C}^m$ if there is no neighborhood U of y such that $F^{-1}(\overline{U}))$ is compact. In other words, F is not proper at y if there is a sequence $x_l \to \infty$ such that $F(x_l) \to y$.

Let S_F denote the set of points at which the mapping F is not proper. The set S_F has the following properties (see [6], [7], [8]):

Theorem 2.2. *Let $X \subset \mathbb{C}^m$ be an irreducible variety of dimension n and let $F = (F_1, ..., F_m) : X \to \mathbb{C}^m$ be a generically-finite polynomial mapping. Then the set S_F is an algebraic subset of $\mathbb{C}^m$ and it is either empty or it has pure dimension $n-1$. Moreover, if $n = m$ then*

$$\deg\ S_F \leq \frac{D(\prod_{i=1}^n \deg\ F_i) - \mu(F)}{\min_{1\leq i\leq n} \deg\ F_i},$$

where D =deg X and $\mu(F)$ denotes the geometric degree of F (i.e., it is a number of points in a generic fiber of F).

The following elementary lemma will be useful in the sequel.

Lemma 2.3. *Assume that a holomorphic curve has parametrization of the type*

$$x(t) = \sum_{-\infty\leq i\leq s} a_i t^i,\ |t| > R,$$

where $t \in \mathbb{C}$, $a_i \in \mathbb{C}^n, \sum_{i>0} |a_i| > 0$ and $s \geq 0$ is an integer. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d. Set

$$\tilde{x}(t) = \sum_{-(d-1)s\leq i\leq s} a_i t^i,\ |t| > R.$$

Assume that $\lim_{t\to\infty} f(x(t)) = b \in \mathbb{C}$, then

$$\lim_{t\to\infty} f(\tilde{x}(t)) = \lim_{t\to\infty} f(x(t)).$$

The same statement holds in the real case.

We also need the following obvious lemma:

Lemma 2.4. *Let $\mathbb{K}$ be an infinite field. Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n. Then a generic linear map $\pi : \mathbb{K}^m \to \mathbb{K}^n$ is finite on X.*

We we state now an effective variant of the curve selection lemma:

Theorem 2.5. *Let $F : \mathbb{C}^n \ni x \to (f_1(x), ..., f_m(x)) \in \mathbb{C}^m$ be a generically finite polynomial mapping. Assume that $\deg f_i = d_i$ and $d_1 \geq d_2 ... \geq d_m$. Let $b \in \mathbb{C}^m$ be a point at which the mapping F is not proper. Then there exists a rational curve with a parametrization of the form*

$$x(t) = \sum_{-(d-1)D-1\leq i\leq D} a_i t^i,\ t \in \mathbb{C}^*,$$

where $a_i \in \mathbb{C}^n, \sum_{i>0} |a_i| > 0$ and $D = \prod_{i=2}^n d_i,\ d = d_1$, such that

$$\lim_{t\to\infty} F(x(t)) = b.$$

Proof. By Lemma 2.4 we can assume that $m = n$. Again by this lemma we can assume that the system of coordinates is sufficiently general. In particular we can assume that the line $l = \{x : x_2 = b_2, x_3 = b_3, ..., x_n = b_n\}$, where $b = (b_1, ..., b_n)$, is not contained neither in the set S_F nor in the set of critical values of F. Let $C = F^{-1}(l)$. Then b is a non proper point of the mapping $F_{|C}$. In particular there exists a holomorphic branch Γ of C such that $lim_{x\in\Gamma, x\to\infty} F(x) = b$. Note that $\deg C \le D$. By Lemma 2.1 we can assume that the branch Γ has a parametrization of the form:

$$x = \sum_{-\infty \le i \le s} a_i t^i, \ |t| > R, \text{ and } \sum_{i>0} |a_i| > 0.$$

From Lemma 2.3 follows that

$$F(\sum_{-(d-1)D-1 \le i \le D} a_i t^i) = b + \sum_{i=1}^{\infty} c_i/t^i,$$

which proves the theorem. $\square$

Definition 2.6. *By a rational arc we mean a curve $\Gamma \subset \mathbb{C}^n$ which has a parametrization $x(t) = \sum_{-D_2 \le i \le D_1} a_i t^i$, $t \in \mathbb{C}^*$, where $a_i \in \mathbb{C}^n$. By a bidegree of the parametrization $x(t)$ we mean a pair of integers (D_1, D_2).*

Definition 2.7. *Let $F = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a generically finite polynomial mapping, which is not proper. Assume that $\deg f_i = d_i$, where $d_1 \ge d_2 ... \ge d_m$. By asymptotic variety of rational arcs of the mapping F we mean the variety $AV(F) \subset \mathbb{C}^{n(2+\prod_{i=1}^n d_i)}$, which consists of those rational arcs $x(t)$ of bidegree (D_1, D_2) where $D_1 = \prod_{i=2}^n d_i$, $D_2 = 1 + (d_1 - 1)\prod_{i=2}^n d_i$, that*

a) $F(x(t)) = b + \sum_{i=1}^{\infty} c_i/t^i$,

b) $\sum_{i>0} \sum_{j=1}^n a_{ij} = 1$, where $a_i = (a_{i1}, ..., a_{in})$.

By generalized asymptotic variety of F we mean the variety $GAV(F) \subset \mathbb{C}^{n(2+\prod_{i=1}^n d_i)}$ defined only by the condition a).

Remark 2.8. The condition b) assures that the arc $x(t)$ "goes to infinity".

Let us note that $AV(F)$ and $GAV(F)$ are algebraic subsets of $\mathbb{C}^{n(2+\prod_{i=1}^n d_i)}$. We identify an arc $x(t)$ with its coefficients $a_{ij} \in \mathbb{C}^{n(2+\prod_{i=1}^n d_i)}$.

Moreover, if $x(t) \in AV(F)$, respectively $x(t) \in GAV(F)$, then $F(x(t)) = \sum c_i(a)t^i$. Note that the function $c_0 : AV(F) \to \mathbb{C}^m$ plays important role:

Proposition 2.9. *Let $c_0 : AV(F) \ni a \mapsto c_0(a) \in \mathbb{C}^m$ be as above. Then*

$$c_0(AV(F)) = S_F.$$

Proof. Let $x(t) = \sum a_i t^i \in AV(F)$. Then $F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(a)/t^i$, this implies that $c_0(a) \in S_F$. Conversely, let $b \in S_F$. By Theorem 2.5 we can find a rational arc $x(t) = \sum_{i=-(d-1)D-1}^{D} a_i t^i$ such that $\lim_{t\to\infty} F(x(t)) = b$. Now change the parametrization of $x(t)$, $t \to \lambda t$ in this way that $\sum_{i>0} \sum_{j=1}^n \lambda^i a_{ij} = 1$. The new arc $x'(t) := x(\lambda t)$ belongs to $AV(F)$ and $c_0(x'(t)) = b$. $\square$

Now let $f \in \mathbb{C}[x_1, ..., x_n]$ be a polynomial. Let us define a polynomial mapping $\Phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^N$ by

$$\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn}),$$

where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$.

Definition 2.10. *Let Φ be as above. Consider the mapping $c_0 : AV(\Phi) \to \mathbb{C}^N$ and the line $L := \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^N$. By a we mean a variety*

$$BV(f) = \{x(t) \in AV(F) : x(t) \in c_0^{-1}(L)\}.$$

Similarly, we define a generalized bifurcation variety of rational arcs of the polynomial f:

$$GBV(f) = \{x(t) \in GAV(F) : x(t) \in c_0^{-1}(L)\}.$$

As an immediate consequence of [9] we have:

Proposition 2.11. *Let $K(f) = K_0(f) \cup K_\infty(f)$ denote the set of generalized critical values of f. If we identify the line $L = \mathbb{C} \times \{(0, ..., 0)\} \subset \mathbb{C} \times \mathbb{C}^N$ with $\mathbb{C}$, then we have $c_0(BV(\Phi)) = K_\infty(f)$ and $c_0(GBV(\Phi)) = K(f)$.*

3. Algorithm

In this section we give an algorithm to compute the set $K_\infty(f)$ of asymptotic critical values as well as the set $K(f)$ of generalized critical values of a complex polynomial f. Let $\deg f = d$ and $D_1 = d^{n-1}, D_2 = d^n - d^{n-1} + 1$.

Algorithm for the set $K_\infty(f)$.

1) Compute equations for the variety $BV(f)$:

a) consider the arc $x(t) = \sum_{-D_2}^{D_1} a_i t^i \in \mathbb{C}^{n(D_1+D_2+1)}$

b) compute $f(x(t)) = \sum c_i(a) t^i$,

c) compute $\frac{\partial f}{\partial x_i}(x(t)) = \sum d_{ik}(a) t^k, i = 1, ..., n,$

d) compute $\frac{\partial f}{\partial x_i}(x(t)) x_j(t) = \sum e_{ijk}(a) t^k, i, j = 1, ..., n$

e) equations for $BV(f)$ are $c_i = 0$ for $i > 0$, $d_{ik} = 0$ for $k \geq 0, i = 1, ..., n$, $e_{ijk} = 0$ for $k \geq 0, i, j = 1, ..., n$ and $\sum_{i>0} \sum_{j=1}^n a_{ij} = 1$, where $a_i = (a_{i1}, ..., a_{in})$.

2) Find equations for irreducible components of $BV(f) = \bigcup_{j=1}^r P_i$. It can be done by standard method of computational algebra. We can use e.g., the MAGMA system and radical decomposition of ideal in this system (see also [1]).

3) Find a point $x_i \in P_i$. It can be also done by standard methods. We can use e.g. the MAGMA system and use several time the elimination procedure in this system. Indeed Let $P_i = V(I)$. Compute $I_k = \mathbb{C}[x_1, ..., x_k] \cap I$ for $k = n, n-1, ...$ until $I_k = (0)$. Then take randomly chosen integer point $(a_1, ..., a_k)$ find a zero $(a_1, ..., a_k, b_1)$ of ideal I_{k+1} and so on.

4) $K_\infty(f) = \{c_0(x_i), i = 1, ..., r\}$.

If we replace above the variety $BV(f)$ by the variety $GAV(f)$ we get an algorithm for computing $K(f)$. Indeed, it is enough to delete at point 1 e) the equation $\sum_{i>0}\sum_{j=1}^{n} a_{ij} = 1$.

4. The real case

We begin with a simple example.

Example 4.1. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ consider $f_{\mathbb{K}} : \mathbb{K}^2 \to \mathbb{K}$, $f(x, y) = x(x^2 + 1)^2$. Observe that $K_\infty(f_{\mathbb{R}}) = K_0(f_{\mathbb{R}}) = \emptyset$. But $0 \in K_\infty(f_{\mathbb{C}}) = K_0(f_{\mathbb{C}})$. So in general*

$$K_\infty(f_{\mathbb{R}}) \neq \mathbb{R} \cap K_\infty(f_{\mathbb{C}}).$$

It shows that the computation of the asymptotic critical values of a real polynomial can not be reduced to the computation of the asymptotic critical values of its complexification.

4.1. **Effective curve selection lemma at infinity.** First we give a construction of a curve selection in a special affine case.

Let $X \subset \mathbb{R}^{2n}$ be an algebraic set described by a system of polynomial equations $p_i = 0$, $\deg p_i \leq d$, where $i = 1, \ldots, n$. Denote by H the hyperplane $\{x_1 = 0\}$. Assume that on $Y := X \setminus H$ the system is non degenerate i.e. $P = (p_1, ..., p_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a submersion at each point of Y. Thus Y is a smooth manifold of dimension n.

Proposition 4.2. *Let $a \in H \cap \overline{Y}$ then there exists an algebraic curve $C \subset \mathbb{R}^{2n}$ of degree $D \leq d^n((d-1)^n + 2)^{n-1}$ such that $a \in \overline{C \cap Y}$.*

Proof. For simplicity we assume that $a = 0$. Denote $\rho(x) = (\sum_{i=1}^{2n} x_i^2)^{\frac{1}{2}}$ and by $S(r)$ the sphere centered at 0 of radius r and finally $Y(r) := Y \cap S(r)$. With our hypothesis we have.

Lemma 4.3. *There exists $\varepsilon > 0$ such that for any $r \in (0, \varepsilon)$ the set $Y(r)$ is a smooth manifold of dimension $n - 1$, in particular is nonempty.*

Indeed, the function ρ restricted to the manifold Y is smooth and semialgebraic, so it has finitely many critical values (see e.g. [3]). Let $\varepsilon_1 > 0$ be the smallest critical value (or $\varepsilon_1 = 1$ if there are no critical values).

On the other hand $\rho|_Y : Y \to \mathbb{R}_+$ is locally trivial (by Hardt's trivialization theorem cf. [3] or [4]), so there is $\varepsilon_2 > 0$ such that for any $r, r' \in (0, \varepsilon_2)$ the sets $Y(r)$ and $Y(r')$ are homeomorphic. Since $a \in H \cap \overline{Y}$, then there is $r' \in (0, \varepsilon_2)$ such that $Y(r')$ is nonempty. Hence $Y(r)$ is nonempty for any $r \in (0, \varepsilon_2)$. Finally we put $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$.

Let us now consider a family of functions $g_\alpha : \mathbb{R}^{2n} \to \mathbb{R}$ of the form $g_\alpha(x) := x_1\alpha(x)$, where $\alpha \in (\mathbb{R}^{2n})^*$ is a linear function on $\mathbb{R}^{2n}$.

Lemma 4.4. *For any $r \in (o, \varepsilon)$ there exists an algebraic set $A_r \subset (\mathbb{R}^{2n})^*$ such that g_α is a Morse function on $Y(r)$, for any $\alpha \notin A_r$. Moreover the set $\bigcup_{r\in(o,\varepsilon)}\{r\} \times A_r$ is contained in a proper algebraic set $A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$.*

The proof of the lemma uses standard arguments in Morse theory, see e.g. [11], [12]. Consider a map

$$\Phi : Y(r) \times (\mathbb{R}^{2n})^* \to (\mathbb{R}^{2n})^*$$

given by $\Phi(x, \alpha) = d_x g_\alpha$. It is enough to show that Φ is a submersion. Indeed the Jacobian matrix of Φ (with respect to variables in $(\mathbb{R}^{2n})^*$) is triangular with x_1 on the diagonal except the entry in the left superior corner where it is $2x_1$. So this matrix is invertible since $x_1 \neq 0$ for $x \in Y(r)$. The second statement follows from the fact that set A_r is defined by polynomial equations with r as a variable parameter.

Lemma 4.5. *There exists $\alpha \in (\mathbb{R}^{2n})^*$ and $0 < \varepsilon' \leq \varepsilon$ such that g_α is a Morse function on each $Y \cap S(r)$, for any $r \in (0, \varepsilon')$.*

Indeed, let $A \subset \mathbb{R} \times (\mathbb{R}^{2n})^*$ by the proper algebraic set in Lemma 4.4. Thus there exists an affine line $\mathbb{R} \times \alpha$ which meets the set A only in finitely many points. So $(0, \varepsilon') \times \{\alpha\}$ is disjoint with A, for some $\varepsilon' > 0$ small enough.

Since $Y(r)$ is not compact so a priori it is not obvious that g_α has a critical point on $Y(r)$. However we have.

Lemma 4.6. *Assume that $Y(r) \neq \emptyset$ and that g_α is Morse on $Y(r)$. Then g_α has a critical point on $Y(r)$.*

Note that image of $Y(r)$ by g_α consists of finitely many nontrivial intervals. Since $\overline{Y(r)}$ is compact and

$$(\overline{Y(r)} \setminus Y(r)) \subset \{x_1 = 0\},$$

Hence at least one of the endpoints of those intervals belongs to $g_\alpha(Y(r))$. Thus g_α achieves a minimum or a maximum in $Y \cap S(r)$.

Proof of Proposition 4.2. Let us fixe a linear form α which satisfies Lemma 4.5. Let Ξ be the locus of critical points of g_α on $Y(r)$ for $r \in (0, \varepsilon')$. The Zariski closure of Ξ is contained in the algebraic set given by the following equations

$$p_1 = \cdots = p_n = 0 \tag{4.1}$$

and

$$dp_1 \wedge \cdots \wedge dp_n \wedge d\rho^2 \wedge dg_\alpha = 0. \tag{4.2}$$

Let us fix Ξ_1 a smooth connected component of Ξ such that $a \in \overline{\Xi}_1$. Then locally Ξ_1 is given by a non degenerate system (4.1) of n equation of degree at most d and $n-1$ equations of degree at most $(d-1)^n + 2$, which are $(n+2) \times (n+2)$ minors of the matrix corresponding to the system (4.2).

Let C be the Zariski closure of Ξ_1. Hence by the general Bezout's formula (cf. e.g. [5] Thm.2.2.5) degree of the curve C is at most $d^n((d-1)^n + 2)^{n-1}$, note that for $d \geq 3$ we have $d^n((d-1)^n + 2)^{n-1} \leq d^{n^2}$.

□

We can state now a real version of Theorem 2.5.

Theorem 4.7. *Let $F : \mathbb{R}^n \ni x \to (f_1(x), ..., f_m(x)) \in \mathbb{R}^m$ be a polynomial mapping. Assume that $\deg f_i \le d$. Let $b \in \mathbb{R}^m$ be a point at which the mapping F is not proper. Then there exists a rational curve with a parametrization of the form*

$$x(t) = \sum_{-(d-1)D-1 \le i \le D} a_i t^i, \ t \in \mathbb{R}^*,$$

where $a_i \in \mathbb{R}^n, \sum_{i>0} |a_i| > 0$ and $D = (d+1)^n (d^n+2)^{n-1}$ such that

$$\lim_{t \to \infty} F(x(t)) = b.$$

Remark 4.8. Note that contrary to the complex case we do not assume that F is generically finite, but the bidegree of the real rational curve is much higher, namely $D = O(d^{n^2})$.

Proof. By Lemma 2.4 we can assume that $m = n$. Let $\gamma(t) \in \mathbb{R}^n$ be a semi-algebraic curve such that $lim_{t\to\infty}|\gamma(t)| = +\infty$ and $lim_{t\to\infty} F(\gamma(t)) = b$. Let $\bar{\gamma}(t)$ be the image of $\gamma(t)$ by the canonical imbedding $\mathbb{R}^n \ni (x_1, \dots, x_n) \to (1 : x_1 : \dots : x_n) \in \mathbb{P}^n$. Since γ is semi-algebraic there exists $a := lim_{t\to\infty}\bar{\gamma}(t) \in H_0$, where H_0 stands for the hyperplane at infinity. Let us denote by Y the graph of F, embedded in $\mathbb{P}^n \times \mathbb{R}^n$ and by X its Zariski closure in $\mathbb{P}^n \times \mathbb{R}^n$. Note that the point (a, b) belongs to the closure (in the strong topology) of Y. Let $\bar{f_i}$ stands for the homogenization of the polynomial f_i, that is

$$\bar{f_i}(x_0, x_1, \dots, x_n) = x_0^{d_i} f_i(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

Hence $X \subset \mathbb{P}^n \times \mathbb{R}^n$ is defined by the equations

$$x_0^{d_i} y_i = \bar{f_i}(x_0, x_1, \dots, x_n),\ i = 1, \dots, n.$$

Assume that $a_1 = 1$, so $\mathbb{R}^n \ni (x_0, x_2, \dots, x_n) \mapsto (x_0, 1, x_2, \dots, x_n) \in \mathbb{P}^n$ is an affine chart around the point a. In this chart X is given by the equations

$$p_i(x_0, x_2, \dots, x_n, y_1, \dots, y_n) := x_0^{d_i} y_i - \bar{f_i}(x_0, 1, x_2, \dots, x_n) = 0,$$

$i = 1, \dots, n$. Clearly $\deg p_i = 1 + d_i$ and $Y = X \setminus \{x_0 = 0\}$ in this chart. So we may apply Proposition 4.2 at the point (a, b). Hence there exists an an algebraic curve $C_1 \subset \mathbb{R}^{2n}$ of degree $d_* \le (d+1)^n (d^n+2)^{n-1}$ such that $(a, b) \in \overline{C_1 \cap Y}$. Now take C the projection of C_1 on $\mathbb{P}^n$. Note that degree of C is less or equal than d_*. Now we can argue as in the proof of Theorem 2.5 to conclude that there is a real rational arc $x(t)$ of bidegree $D_2 = (d-1)D + 1$, $D_1 = D$ where $D = (d+1)^n (d^n+2)^{n-1}$, such that $lim_{t\to\infty} F(x(t)) = b$.

□

Definition 4.9. *By a real rational arc we mean a curve $\Gamma \subset \mathbb{R}^n$ which has a parametrization $x(t) = \sum_{-D_2 \le i \le D_1} a_i t^i,\ t \in \mathbb{R}^*$, where $a_i \in \mathbb{C}^n$. By bidegree of Γ we mean a pair of integers (D_1, D_2).*

Definition 4.10. *Let $F = (f_1, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a generically finite polynomial mapping, which is not proper. Assume that $\deg f_i \le d$ and $D = (d+1)^n (d^n + 2)^{n-1}$ By asymptotic variety of real rational arcs of the mapping F we mean the*

variety $AV_{\mathbb{R}}(F) \subset \mathbb{R}^{n(dD+2)}$, which consists of those real rational arcs $x(t)$ of bidegree $(D, (d-1)D+1)$ that

a) $F(x(t)) = b + \sum_{i=1}^{\infty} c_i/t^i$,

b) $\sum_{i>0} \sum_{j=1}^{n} a_{ij}^2 = 1$, where $a_i = (a_{i1}, ..., a_{in})$.

If we omit the condition b) we get definition of a generalized asymptotic variety of real arcs $GAV_{\mathbb{R}}(F)$.

Remark 4.11. The condition b) assures that the arc $x(t)$ "goes to infinity".

As before we see that $AV_{\mathbb{R}}(F), GAV_{\mathbb{R}}$ are algebraic subsets of $\mathbb{R}^{n(dD+2)}$. Moreover, for $x(t) \in AV_R(F)(GAV_{\mathbb{R}}(F))$ we have $F(x(t)) = \sum c_i(a)t^i$. Note that again the function $c_0 : AV_{\mathbb{R}}(F) \to \mathbb{R}^m$ plays important role:

Proposition 4.12. *Let $c_0 : AV_{\mathbb{R}}(F) \in a \mapsto c_0(a) \in \mathbb{R}^m$ be as above. Then*

$$c_0(AV_{\mathbb{R}}(F)) = S_F(\mathbb{R}).$$

Proof. Let $x(t) = \sum a_i t^i \in AV_{\mathbb{R}}(F)$. Then $F(x(t)) = c_0(a) + \sum_{i=1}^{\infty} c_i(a)/t^i$, this implies that $c_0(a) \in S_F$. Conversely, let $b \in S_F$. By Theorem 4.7 we can find a rational arc $x(t) = \sum_{i=-(d-1)D-1}^{D} a_i t^i$ such that $\lim_{t\to\infty} F(x(t)) = b$. There exist a $\lambda \in \mathbb{R}$ such that $\sum_{i>0} \sum_{j=1}^{n} \lambda^{2i} a_{ij}^2 = 1$. Change a parametrization by $t \to \lambda t$. The new arc $x'(t) := x(\lambda t)$ belongs to $AV_{\mathbb{R}}(F)$ and $c_0(x'(t)) = b$. $\square$

Now let $f \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial. Let us define a polynomial mapping $\Phi : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^N$ by

$$\Phi = (f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, ..., h_{nn}),$$

where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i = 1, 2..., n$, $j = 1, ..., n$.

Definition 4.13. *Let Φ be as above. Consider the mapping $c_0 : AV_{\mathbb{R}}(\Phi) \to \mathbb{R}^N$ and the line $L := \mathbb{R} \times \{(0, ..., 0)\} \subset \mathbb{R} \times \mathbb{R}^N$. By a bifurcation variety of real rational arcs of the polynomial f we mean a variety*

$$BV_{\mathbb{R}}(f) = \{x(t) \in AV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L)\}.$$

Similarly we define

$$GBV_{\mathbb{R}}(f) = \{x(t) \in GAV_{\mathbb{R}}(\Phi) : x(t) \in c_0^{-1}(L)\}.$$

As an immediate consequence of [9] we have:

Proposition 4.14. *Let $K(f)(\mathbb{R}) = K_0(f) \cup K_\infty(f)$ denote the set of generalized critical values of real polynomial f. If we identify the line $L = \mathbb{R} \times \{(0, ..., 0)\} \subset \mathbb{R} \times \mathbb{R}^N$ with $\mathbb{R}$, then we have $c_0(BV_{\mathbb{R}}(f)) = K_\infty(f)$ and $c_0(GBV_{\mathbb{R}}(f)) = K(f)$.*

5. Real algorithm

In this section we describe an algorithm to compute the set $K_\infty(f)$ of asymptotic critical values as well as the set $K(f)$ of generalized critical values of a real polynomial $f \in \mathbb{R}[x_1, ..., x_n]$. Let $\deg f = d$ and $D_1 = (d+1)^n(d^n+2)^{n-1}, D_2 = (d-1)D_1+1$.

Algorithm for the set $K_\infty(f)$.

1) Compute equations g_α for the variety $BV_{\mathbb{R}}(f)$:

a) consider the arc $x(t) = \sum_{-D_2}^{D_1} a_i t^i \in \mathbb{R}^{n(D_1+D_2+1)}$

b) compute $f(x(t)) = \sum c_i(a)t^i$,

c) compute $\frac{\partial f}{\partial x_i}(x(t)) = \sum d_{ik}(a)t^k, i = 1, 2,$

d) compute $\frac{\partial f}{\partial x_i}(x(t))x_j(t) = \sum e_{ijk}(a)t^k, i, j = 1, 2$

e) equations for $BV_{\mathbb{R}}(f)$ are $c_i = 0$ for $i > 0$, $d_{ik} = 0$ for $k \geq 0, i = 1, ..., n$, $e_{ijk} = 0$ for $k \geq 0, i, j = 1, ..., n$ and $\sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1$, where $a_i = (a_{i1}, ..., a_{in})$.

2) Form a polynomial $G = \sum_\alpha g_\alpha^2$, where g_α are c_i for $i > 0$, or d_{ik} for $k \geq 0, i = 1, ..., n$, or e_{ijk} for $k \geq 0, i, j = 1, ..., n$ or $\sum_{i>0} \sum_{j=1}^n a_{ij}^2 - 1$.

3) In each connected component S_i of the set $G = 0$ find a point $x_i \in S_i, i = 1, ...r$. It can be done by standard method of computational algebra, e.g. Theorem 15.13 p. 585 in [2].

4) $K(f) = \{c_0(x_i), i = 1, ..., r\}$.

If we replace above the variety $BV_{\mathbb{R}}(f)$ by the variety $GAV_{\mathbb{R}}(f)$ we get an algorithm for computing $K(f)$. Actually it is enough to delete at points 1 e) and 2) the equation $\sum_{i>0} \sum_{j=1}^n a_{ij}^2 = 1$.

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